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THE CONTROLLABILITY OF NON-HOLONOMIC MECHANICAL SYSTEMS WITH CONSTRAINED CONTROLS†

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A cycle of investigations, related to the problem of the controllability of non-linear dynamical systems, is developed. Systems which have a mechanical nature (wheeled means of transportation, transport and machining systems, and manipulators) are considered. Controllability criteria are established for the general class of mechanical systems which may contain non-holonomic constraints; similar criteria were established previously for holonomic systems in [1–4]. The controllability conditions obtained have a clear physical meaning. For example, for the controllability of a manipulation robot it is necessary that the control forces should predominate over the other generalized forces (the weight forces, and the resistance forces due to the external medium), Predominance is necessary as regards the amplitude of the forces. Additional conditions are related to the properties of constraints imposed on the system. This essentially requires that the constraint relations allow of the possibility of an appropriate change in the coordinates and velocities of the mechanical system in the region investigated.

1. OBJECT OF THE INVESTIGATION

The object of the investigation is dynamical systems, the motion of which are described by Lagrange equations of the second kind [5–7]

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = Q_i + M_i + R_i, \quad R_i = \sum_{s=1}^{g} \Lambda_s f_{si} \tag{1.1}$$

$$\sum_{i=1}^{N} f_{si}(q)\dot{q}_i = 0 \tag{1.2}$$

Here and henceforth the subscripts take the following values: $i, j = 1, 2, \ldots, N$; $s = 1.2, \ldots, g$; $r = g+1, g+2, \ldots, N$.

System (1.1), (1.2) describes the motion of many mechanical systems. It can describe the motion of holonomic systems in dependent coordinates, if, for some reasons, it is desirable to take into account the description of the constraints (1.2) in explicit form [5–7], for example, when the replacement of the initial generalized coordinates leads to a loss in clarity or expressiveness of the control problem in question or when the reactions of the constraints imposed are being investigated. System (1.1), (1.2) can also describe the motion of non-holonomic systems if relations (1.2) describe non-holonomic mechanical constraints. The topic under discussion is, primarily, systems with rolling: automobiles, trains, aircraft on a take-off strip, etc., and also electromechanical systems with sliding contacts [5–13].

We will use the following standard notation: q_i and $\dot{q}_i$ are the generalized coordinates and velocities of the system, N is the number of coordinates, R_i is the reaction of the constraint (1.2), Λ_s are Lagrange undetermined coefficients, and $\{Q_i + M_i\}$ are generalized forces. The quantities M_i are considered as control forces (controls), which are produced by the control devices of the system – the drives. The

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quantities $Q_i = Q_i(q, \dot{q}, t)$ are determined by external forces, which act on the mechanical system. We will denote by $T = T(q, \dot{q})$ the kinetic energy of the system

$$T = \frac{1}{2}\sum_{i,j=1}^{N} a_{ij}(q)\dot{q}_i\dot{q}_j \tag{1.3}$$

The quantities a_{ij} are related to the mass distribution of the mechanical system (1.1) and characterize its inertial properties. The following inequalities, well known in mechanics, hold for the kinetic energy T

$$\lambda_1|\dot{q}|^2 \le T \le \lambda_2|\dot{q}|^2, \quad |\dot{q}|^2 = \sum_{i=1}^{N} \dot{q}_i^2, \quad \lambda_p = \text{const} \ge 0 \tag{1.4}$$

for any values of q_i and $\dot{q}_j$.

For a formal analysis of the controllability property of system (1.1) we will introduce the following assumptions regarding the properties of the object being investigated. We will assume that, in inequalities (1.4),

$$0 < \lambda_1 \le \lambda_2 < \infty \tag{1.5}$$

The values of the force Q_i in system (1.1) for all q_i, $\dot{q}_j$ and $t \ge t^0$ (t^0 is a certain instant of time) will be assumed to be bounded:

$$|Q_i(q, \dot{q}, t)| \le h_i, \quad h_i = \text{const} \ge 0 \tag{1.6}$$

The linear formulae (1.2) are assumed to be linearly independent

$$\text{rank}\,\|f_{si}\| = g \tag{1.7}$$

This assumption is assumed to be natural in the dynamics of systems with non-holonomic constraints.

We consider as the *permissible* controls in system (1.1) the time functions $M(t) = \{M_1(t), \dots, M_N(t)\}$, the values of which for all t are bounded:

$$|M_i(t)| \le H_i, \quad H_i = \text{const} \ge 0 \tag{1.8}$$

We will denote the class of such functions by $U = U(H_i)$.

We will consider the functions of time $q = q(t)$, $t > t^0$ with absolutely continuous derivative as the solutions of system (1.1), (1.2). The functions $M_i(t)$ and $Q_i(q, \dot{q}, t)$ in this set of solutions are assumed to be summable in any finite time interval. The functions $a_{ik}(q)$, $\partial a_{ij}(q)/\partial q_k$ and $f_{ik}(q)$ are assumed to be continuous. The formal assumptions introduced are fairly natural [1–13].

2. FORMULATION OF THE PROBLEM

The property of controllability of systems of the form (1.1), (1.2) will be understood in Kalman's sense.

Definition 1. System (1.1), (1.2) is said to be controllable in its state space $\{q_1, \dots, q_N, \dot{q}_1, \dots, \dot{q}_N\}$ in the class of permissible controls U, if, for arbitrary points $S^f = (q^f, \dot{q}^f)$ and $S^e = (q^e, \dot{q}^e)$ of space, a certain permissible control $M(t) \in U$ exists, for which system (1.1), (1.2) can transfer from S^f to S^e in a certain finite time.

The controllability conditions were obtained previously in [1–3] for holonomic mechanical systems. The constraints (1.2) are ignored, system (1.1), (1.2)

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = Q_i + M_i \tag{2.1}$$

will be such a system. The condition for system (2.1) to be controllable has the form

$$H_i > h_i \tag{2.2}$$

The numbers h_i and H_i were introduced in relations (1.6) and (1.8). In other words, it is required that the controls M_i of the mechanical system should predominate over the generalized forces Q_i in amplitude. Such conditions are regarded as natural. If the weight of the load exceeds the lifting power of the manipulator, it is obviously difficult to control such a system.

Note that, if constraints (1.2) are holonomic, system (1.1), (1.2) can also be reduced to a system of type (2.1). The conditions for such a system to be controllable are similar to conditions (2.2). Note also that constraints (1.2) are holonomic if they can be reduced to the form of geometrical constraints. The point here is that certain systems of differential equations (1.2) can be integrated and its integrals can be written in the form of the relations $\sum \tilde{f}_{si}(q) = 0$, which contain only coordinates of the system. Examples of constraints (1.2) exist when this is impossible (such constraints are called non-holonomic constraints).

In general, system (1.1), (1.2) is not necessarily a holonomic mechanical system. For a non-holonomic system, the well-known criteria of controllability are not directly applicable. The problem considered in this paper is to obtain the conditions for mechanical systems of the form (1.1), (1.2) to be controllable in the class U of bounded controls.

3. THE SUBSYSTEM OF CONSTRAINTS

In investigating the problem of the controllability of system (1.1), (1.2), the properties of its subsystem (1.2), i.e. the subsystem describing the mechanical constraints, play an important role. These properties may essentially determine the possibility of controlling the initial system (1.1), (1.2). In fact, the subsystem of constraints (1.2) may contain an element, for example, of the form $\dot{q}_1 = 0$. In this case $q_1(t) \equiv q_i(0)$, $t \geq 0$, and system (1.2) cannot be transferred to the point q_1^* of phase space, for which $q_1^* \neq q_1(0)$. Consequently, system (1.1), (1.2) is not controllable. This example is formal, but the relation $\dot{q}_1 = 0$ corresponds to all the indicators for describing a certain mechanical constraint (this example is given only for simplicity). The overall aim of this paper is to draw attention to this kind of circumstance and to develop an appropriate method of investigating the controllability problem.

Thus, we will consider the subsystem of constraints (1.2)

$$\sum_{i=1}^{N} f_{si}(q)\dot{q}_i = 0, \quad s = 1, 2, \dots, g \tag{3.1}$$

as a certain independent system of differential equations. The state space of system (3.1) has the form $\{q_1, \dots, q_N\}$, where N is the dimension of the space. In system (3.1) there are g equations, where $N > g$. The last inequality is a consequence of the natural assumption that the initial mechanical system has at least one degree of freedom, i.e. $n \geq 1$, where $n = N - g$.

Hence, in system (3.1) the number of variables N is greater than the number of equations g. Therefore, systems of the form (3.1) may possess the following feature: several solutions of system (3.1) can pass through one and the same point of state space $\{q_1, \dots, q_N\}$ (see the specific example in Section 7). Let us say that different solutions $q^p(t) = (q_1^p(t), \dots, q_N^p(t))$, $t \geq t^1$ $(p = 1, 2, \dots)$ of system (3.1) can pass through a certain point $s^+ = (q_1^+, \dots, q_N^+)$, i.e.

$$q^p(t^1) = s^+, \quad p = 1, 2, \dots \tag{3.2}$$

Then, we can introduce the following set for the points s^+

$$Z(s^+, \tau) = \{q^1(t^1 + \tau), q^2(t^1 + \tau), \dots\} \tag{3.3}$$

The elements of this set are the points $q^1(t^1 + \tau), q^2(t^1 + \tau), \dots$ of the state space $\{q_1, \dots, q_N\}$ of system (3.1). The set $Z(s^+, \tau)$ is similar to the attainability set, which has meaning for controllable dynamical systems [14, 15]. In this connection, we will introduce the following definition.

Definition 2. The point $s^- = (q_1^-, \dots, q_N^-)$ of the state space $\{q_1, \dots, q_N\}$ of system (3.1) will be said to be attainable from the point $s^+ = (q_1^+, \dots, q_N^+)$ if a permissible trajectory $q^*(t)$ of motion of system (3.1) exists such that $0 \leq \tau < \infty$, where $q^*(t^1) = s^+$, $q^*(t^1 + \tau) = s$.

In Definition 2 we are essentially dealing with the following property of system (3.1). Suppose we are given two points, s^+ and s^-, for system (3.1). Suppose that, at a certain initial instant of time t^1, system

(3.1) is situated at the point s^+, i.e. $(q_1(t^1), \dots, q_N(t^1)) = s^+$. Suppose also that, at a certain instant of time $t = t^1 + \tau$, the set (3.3) $Z(s^+, \tau)$ contains the point s^-. If $\tau < \infty$, the point s^- can be attained from the point s^+, by Definition 2. Hence, Definition 2 indicates the fact that certain points of the space $\{q_1, \dots, q_N\}$ can be connected by finite sections of the trajectory of system (3.1). It turns out that this fact is important later for analysing the property of controllability of the initial mechanical system.

Definition 3. The functions of time $q(t) = (q_1(t), \dots, q_N(t))$, for which the derivatives $\ddot{q}_j(t)$ are continuous will be assumed to be permissible trajectories in Definition 2.

The introduction of this class of permissible trajectories is due to the fact that it will be proposed later that the solutions $q(t)$ of system (3.1) should be considered as solutions of initial system (1.1), (1.2). From this point of view not all solutions of system (3.1) will be of interest, but only fairly smooth ones. In particular, it is necessary here to take into account constraints (1.6) and (1.8), imposed on the controls and the generalized force Q_i and M_i of the initial mechanical system. Hence, we will further consider those solutions $q(t)$ of system (3.1) for which the derivatives $\ddot{q}_j(t)$ are continuous.

We will formulate the following proposition, which is the basis of our investigation of the property of controllability of the initial system (1.1), (1.2).

The property of attainability. At point $s^- = (q_1^-, \dots, q_N^-)$ of the state space $\{q_1, \dots, q_N\}$ of system (3.1) is attainable from its arbitrary point $s^+ = (q_1^+, \dots, q_N^+)$ in the class of permissible trajectories.

An interesting example of system (3.1), which possesses the property of attainability is investigated in Section 7. It is clear intuitively that the property described is important for solving the problem of the controllability of the initial system (1.1), (1.2). Namely, suppose the subsystem of constraints (1.2) does not possess the property of attainability, i.e. suppose a certain point s^- is not attainable from a certain point s^+. Then system (1.1), (1.2) will not, in general, possess the property of controllability.

4. BASIC RESULT

Theorem 1. Suppose conditions (1.3)–(1.8) are satisfied for system (1.1), (1.2). Suppose we are given arbitrary constants H_i which satisfy the condition

$$H_i > h_i, \quad 1 = 1, 2, \dots, N \tag{4.1}$$

Suppose the property of attainability holds for subsystem (1.2) of system (1.1), (1.2). Then system (1.1), (1.2) is controllable in its phase space $\{q_1, \dots, q_N, \dot{q}_1, \dots, \dot{q}_N\}$ in the class of controls $U(H_i)$.

The proof of Theorem 1 is given in Section 5.

The meaning of Theorem 1 is that the mechanical system (1.1), (1.2) can be transferred to any point of the phase space of the system. It is of no importance where the system was situated at the initial instant of time. For this it is sufficient that the control forces M_i, $|M_i| \le H_i$ of the system should predominate over the forces Q_i, $|Q_i| \le h_i$, by condition (4.1). Theorem 1 was previously stated in [1, 2] without taking into account the imposed constraints (1.2) (note that condition (4.1) is identical with the condition (2.2)). Unlike this, in Theorem 1 the controllability of a mechanical system is based on the proposition that these constraints are allowed. It is only necessary that, for the subsystem of constraints (1.2), the property of attainability should also be satisfied. In addition, we note the following. Suppose conditions (2.2) are satisfied for system (2.1), and it is controllable. Then system (2.1) remains controllable if constraints (1.2) are imposed on it. This will be so if the constraints possess the property of attainability.

The question of how far condition (4.1) of Theorem 1 is from the necessary conditions is of interest. We will show that when condition (4.1) breaks down, system (1.1), (1.2) may be uncontrollable. Thus, suppose instead of conditions (4.1) the following conditions hold

$$H_i = h_i \tag{4.2}$$

We will consider a special case of system (1.1), (1.2), where $Q_i = 0$ and, consequently, $h_i = H_i = 0$. Hence we have the relations $\dot{T}(t) = 0$, $t \ge 0$ (in deriving which one can take into account the proof of Lemma 1 given below). Consequently, system (1.1), (1.2) cannot be transferred to a point in phase space, where the generalized velocities of the system are fairly high. The system is therefore not controllable.

The usefulness of Theorem 1 is illustrated by an example in Section 7.

5. PROOF OF THEOREM 1: REDUCTION TO THEOREM 2 AND THEN TO THEOREM 3

The basis of the proof of Theorem 1 is the scheme developed by Pyatnitskii [1, 2].

Consider the system

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = m_i(t) + R_i, \quad i = 1, 2, \dots, N,$$
$$\sum_{i=1}^{N} f_{si}(q)\dot{q}_i = 0, \quad s = 1, 2, \dots, g \tag{5.1}$$

The initial system (1.1), (1.2) converts into it if the required controls M_i are chosen in the form

$$M_i(t) = -Q_i(q(t), \dot{q}(t), t) + m_i(t) \tag{5.2}$$

The quantities $m_i(t)$ will be considered as new controls, and we will introduce a class of such controls.

Class u of the controls includes functions of time $m_i(t)$ of the form (5.2), which satisfy the conditions

$$|m_i(t)| \le H_i^0, \quad H_i^0 = \text{const} > 0 \tag{5.3}$$

Suppose system (5.1) is controllable in the class u. We will show that the initial system (1.1), (1.2) is then controllable in the class U.

In fact, suppose a certain control $m^*(t)$ exists which transfers system (5.1) from an arbitrary point S^f to an arbitrary point S^e in a certain finite time along a certain trajectory $q = q^*(t)$. Then, obviously, the control

$$M_i^*(t) = -Q_i(q^*(t), \dot{q}^*(t), t) + m_i^*(t) \tag{5.4}$$

will also transfer system (1.1), (1.2) from the point S^f to the point S^e in a finite time.

Control (5.4) will belong to the class U. For this, we will choose positive numbers H_i in inequalities (5.3) from the condition

$$H_i^0 = H_i - h_i \tag{5.5}$$

In addition, we will assume that relations (5.4) are satisfied when

$$t \ge t^0 \tag{5.6}$$

The last condition is related to the fact that we have taken (1.6) into account, which is satisfied for the initial system (1.1), (1.2). Hence, the assertion of Theorem 1 follows from the following assertion.

Theorem 2. Suppose we are given an arbitrary system (5.1), which satisfies conditions (1.3)–(1.7). Suppose the property of attainability is satisfied for system (5.1). Suppose also that certain positive numbers H_i^0, which define the class $u(H_i^0)$ of permissible controls, are specified. Suppose we are given two arbitrary points, S^f and S^e, of the phase space of system (5.1). Then, a certain control from the class u exists, which when $t \ge t^0$ transfers system (5.1) from the points S^f to point S^e in a certain finite time.

In a special case, Theorem 2 has the form of the following theorem.

Theorem 3. Suppose we are given an arbitrary system (5.1), which satisfies conditions (1.3)–(1.8). Suppose the attainability property is satisfied for system (5.1). Suppose also that we are given certain positive number H_i^0. Suppose we are given an arbitrary point S^f of the phase space of system (5.1). Then, a control from the class $u(H_i^0)$ exists, which transfers system (5.1) from the point S^f to the origin of coordinates $S^0 = (0, 0)$ in a certain finite time.

Theorem 2 is a corollary of Theorem 3. In fact, by Theorem 3 a control exists which transfers system (5.1) from S^f to the origin of coordinates $S^0 = (0, 0)$ in a certain finite time. The control which transfers the system from the origin of coordinates to the point S^e also exists.

In fact, in system (5.1) we change the direction of time, i.e. we make the replacement

$$t = t^1 - \theta \tag{5.7}$$

We then obtain the system

$$\frac{d}{d\theta}\frac{\partial T}{\partial q_i'} - \frac{\partial T}{\partial q_i} = m_i(t^1 - \theta) + R_i, \quad \sum_{i=1}^{N} f_{si}(q) q_i' = 0 \tag{5.8}$$

The expressions for the kinetic energy T and the reactions of the constraints are similar to expressions (1.3) and the second formula of (1.1), respectively, where $q' = dq/d\theta$. In (5.7) we have denoted the constant, to be determined, by t^1. System (5.8), like system (5.1), satisfies conditions (1.3)–(1.7) (with the same parameters λ_1, λ_2 and h_i), and the attainability property holds for it. Hence, all the conditions of Theorem 3 are satisfied for system (5.8).

This means that system (5.8) can be transferred from an arbitrary point S^e to the origin of coordinates S^0 in a certain finite time. We will denote the corresponding control from the class u by $m_i^1(\theta)$. It is then obvious that a control of the from $m_i^{11}(t) = m_i^1(t^1 - t)$ will transfer system (5.1) from the origin of coordinates to the point S^e also in a finite time.

We will show that the control $m_i^{11}(t)$ will transfer system (5.1) from the point S^0 to the point S^e when $t \geq t^0$, which is required in Theorem 2. In order to show this, we choose the constant t^1 in (5.7) in an appropriate way. Namely, suppose system (5.1) when $t = t^0$ begins its motion from the point S^f, and at a certain instant of time $t = t^{00}$ is incident on the point S^0. Similarly, system (5.8) is incident from the point S^e when $\theta = \theta^0$ on the point S^0 at a certain instant of time $\theta = \theta^{00}$. We will put $t^1 = t^{00} + \theta^{00}$ in Eq. (5.7). Then the control $m_i^1(\theta)$ will correspond to the segment $[\theta^0, \theta^{00}]$, and the control $m_i^{11}(t)$ will correspond to the segment $[t^{00}, t^{00} + (\theta^{00} - \theta^0)]$. Consequently, the control $m_i^{11}(t)$ begins to transfer system (5.1) from S^0 to S^e at the required instant of time $t = t^{00} \geq t^0$. Thus, Theorem 2 follows from Theorem 3.

6. PROOF OF THEOREM 3

The proof of Theorem 3 rests on the following two main properties of mechanical systems of the form (5.1): (1) system (5.1) can be slowed down completely and transferred into the coordinate plane; (2) system (5.1) can be shifted from one point of the coordinate plane to another.

In Fig. 1 the control m^f stops system (5.1), and the control m^{f0} transfers it from a point in the coordinate plane to the origin of coordinates.

The possibility of slowing down system (5.1) (as a result of the control m^f) and the possibility of displacing system (5.1) from one point of the coordinate plane to another (by means of control m^{f0}) are determined by the following lemmas. These lemmas complete the proof of Theorem 3.

Lemma 1. Suppose the conditions of Theorem 3 are satisfied. Then a certain control m^f of the class u exists, which transfers system (5.1) in a finite time from an arbitrary initial point $S^f = (q^f, \dot{q}^f)$ to a certain point of the form $S^{f0} = (q^{f0}, 0)$.

Lemma 2. Suppose the conditions of Theorem 3 are satisfied. Then a certain control $m^{f0}(t)$ of the class u exists, which in a finite time transfers system (5.1) from an arbitrary point of the form $S^{f0} = (q^{f0}, 0)$ to the origin of coordinates $S^0 = (0, 0)$.

Proof of Lemma 1. Consider a mechanical system of the form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = -H_i^0 \operatorname{sign}(\dot{q}_i) + R_i, \quad \sum_{i=1}^{N} f_{si}\dot{q}_i = 0, \quad t \geq 0 \tag{6.1}$$

We convert mechanical system (5.1) into system (6.1) using the control

$$m_i = m_i^f = -H_i^0 \operatorname{sign}(\dot{q}_i) \tag{6.2}$$

(this control is m^f in Fig. 1). The controls (6.2) are permissible, i.e. they belong to the class u. In system (6.1) the initial conditions $q(0) = q^f$, $\dot{q}(0) = \dot{q}^f$ are arbitrary, in accordance with the conditions of Theorem 3. The lemma is true if the following relations holds for the motions of system (6.1)

$$|\dot{q}_i(\tau)| = 0, \quad \exists \tau: 0 \leq \tau < \infty \tag{6.3}$$

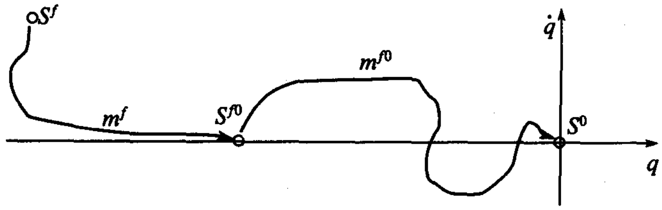

Fig. 1

In fact, the following relations, well known in mechanics, are satisfied during the motions of system (6.1)

$$\dot{T} = \sum_{i=1}^{N} \dot{q}_i \{m_i^f + R_i\} \tag{6.4}$$

The following equation holds for the reactions R_i at any instant of time

$$\sum_{i=1}^{N} \dot{q}_i R_i = 0 \tag{6.5}$$

In fact, it follows from the assumption of ideality of constraints imposed on system (6.1) that

$$\sum_{i=1}^{N} \delta q_i R_i = 0 \tag{6.6}$$

By definition, equality (6.6) holds for any vector $(\delta q_1, \dots, \delta q_N)$, which satisfies the system

$$\sum_{i=1}^{N} f_{si} \delta q_i = 0 \tag{6.7}$$

The vector $(\dot{q}_1, \dots, \dot{q}_N)$ in Eq. (6.5) satisfies system (6.7), since it satisfies the description of the constraints in system (6.1). Consequently, Eq. (6.6), i.e. Eq. (6.5), holds for the vector $(\dot{q}_1, \dots, \dot{q}_N)$.

When relations (6.5) and (6.2) are taken into account, Eq. (6.4) can be written in the form

$$\dot{T} = \sum_{i=1}^{N} \dot{q}_i \{-H_i^0 \operatorname{sign}(\dot{q}_i)\}$$

This yields the following relations

$$\dot{T} = -\sum_{i=1}^{N} |\dot{q}_i| H_i^0 \le -\lambda\sqrt{T}, \quad \lambda = \text{const} > 0 \tag{6.8}$$

where we have taken into account inequalities (1.4) and (1.5) and the inequality $H_i^0 > 0$. The following relation holds for solutions of differential inequality (6.8)

$$T(t) = 0, \text{ when } t \ge \tau, \quad \tau = 2\sqrt{T(0)}/\lambda \tag{6.9}$$

Hence, taking (1.4) into account, the required equalities (6.3) follow.

Proof of Lemma 2. We will write the initial system (5.1) in the equivalent form of Magg's equations [5]

$$\sum_{i=1}^{N}(L_i - m_i)b_{ir} = 0, \quad r = g+1, \ldots, m,$$
$$\sum_{i=1}^{N} f_{si}\dot{q}_i = 0, \quad L_i = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} \tag{6.10}$$

We have denoted certain functions $b_{ir} = b_{ir}(q)$ by b_{ir}, where

$$\text{rank}\|b_{ir}\|_{i=1,N}^{r=g+1,N} = n, \quad n = N - g \geq 1 \tag{6.11}$$

Remarks. 1. To construct system (6.10), the original equations (5.1) will be written initially in the general form of d'Alembert–Lagrange equations

$$\sum_{i=1}^{N}(L_i - m_i)\delta q_i = 0 \tag{6.12}$$

(we have taken into account the fact that constraints (1.2) are assumed to be ideal). We will further take into account the fact that the virtual displacements δq_i in relation (6.12) satisfy g equations (6.7). Consequently, the virtual displacement δq_i can be expressed in terms of certain $n = N - g$ independent parameters e_r [5]

$$\delta q_i = \sum_{r=g+1}^{N} b_{ir}e_r$$

This enables Eqs (6.10) to be obtained from Eqs (6.12).

2. The functions $b_{ir} = b_{ir}(q)$ in system (6.10) can be constructed in explicit form as follows. Consider the system

$$\sum_{i=1}^{N} f_{si}(q)\dot{q}_i = 0, \quad s = 1, \ldots, g, \quad \sum_{i=1}^{N} f_{ri}(q)\dot{q}_i = \dot{\pi}_r, \quad r = g+1, \ldots, N \tag{6.13}$$

The first group of relations in system (6.13) is identical with the descriptions of constraints (1.2). The second group of relations is introduced in addition. Here $\dot{\pi}_r$ $(r = g + 1, g + 2, \ldots, N)$ are certain quantities (quasi-velocities in the corresponding system of Appell's equations). The relations of the second group define the functions $f_{ri}(q)$ $(r = (g + 1, \ldots, N))$, on which the following non-restrictive condition is imposed [5, 6]

$$\text{rank}\|f_{ri}\|_{r=1,N}^{i=1,N} = N \tag{6.14}$$

3. An example of a specific choice of the functions $f_{rj}(q)$ is given in Section 7. The functions $b_{ir} = b_{ir}(q)$ in system (6.10) can be connected with the functions $f_{ri}(q)$ by the relation

$$BF = E, \quad B = \|b_{ij}\|_{j=1,N}^{i=1,N}, \quad F = \|f_{ij}\|_{j=1,N}^{i=1,N}, \quad \text{rank}F = N \tag{6.15}$$

Where E is the identity matrix.

Lemma 2 follows from the following assertion: functions $q^* = q^*(t)$ and $m^{f0} = m^{f0}(t)$ exist, such that $q^*(t)$, $0 \leq t \leq \tau < \infty$ is the solution of system (6.10) for the control $m = m^{f0}(t)$, where

$$|m_i^{f0}(t)| \leq H_i^0, \quad i = 1, 2, \ldots, N, \quad 0 \leq t \leq \tau \tag{6.16}$$

$$q^*(0) = q^{f0}, \quad \dot{q}^*(0) = 0, \quad q^*(\tau) = 0, \quad \dot{q}^*(\tau) = 0 \tag{6.17}$$

where q^{f0} is an arbitrary specified vector.

Proof of the assertion. A certain function $q^* = q^*(t)$, $0 \leq t \leq \tau$ will be a solution of system (6.10) for a certain control $m^{f0} = m^{f0}(t)$, if relations of the form

$$\sum_{i=1}^{N} L_i|_{q=q^*(t)} b_{ir}(q^*) \equiv \sum_{i=1}^{N} m_i^{f0}(t)b_{ir}(q^*), \quad r = g+1, \ldots, N$$

hold as well as inequalities (6.16). These relations and inequalities hold if the inequalities

$$\sum_{i=1}^{N} |L_i| |_{q=q^*(t)} |b_{ir}(q^*)| \le \sum_{i=1}^{N} H_i^0 |b_{ir}(q^*)|, \quad 0 \le t \le \tau$$

or the inequalities

$$|L_i| |_{q=q^*(t)} \le H_i^0, \quad 0 \le t \le \tau$$

hold, which we will write in expanded form

$$\left| \sum_{j=1}^{N} a_{ij}\ddot{q}_j + \sum_{j,p=1}^{N} \left[\frac{\partial a_{ij}}{\partial q_p} - \frac{1}{2}\frac{\partial a_{jp}}{\partial q_i} \right] \dot{q}_p \dot{q}_j \right| \Bigg|_{q=q^*(t)} \le H_i^0, \quad 0 \le t \le \tau \tag{6.18}$$

We will show that inequalities (6.18) are a consequence of the property of attainability for constraints, which ae assumed to hold in Theorem 3.

In fact, according to the property of attainability, a certain function $q^1 = q^1(t)$ exists, which satisfies the constraints (1.2) when $0 \le t \le \tau^1$. The function $q^1 = q^1(t)$ also satisfies the specified initial conditions (6.17) of the form

$$q^1(0) = q^{f0}, \quad q^1(\tau^1) = 0$$

where the functions $\ddot{q}_i^1(t)$ are continuous.

The following limits follow from the property of continuity of the function $\ddot{q}_i^1(t)$ in the interval $0 \le t \le \tau^1$

$$\begin{gathered} |q_i^1(t)| \le D^0, \quad |\dot{q}_i^1(t)| \le D^1, \quad |\ddot{q}_i^1(t)| \le D^2, \\ i = 1, 2, \ldots, N, \quad 0 \le t \le \tau^1; \quad D^p = \text{const} \ge 0 \end{gathered} \tag{6.19}$$

We will further take into account the fact that the functions $a_{ik}(q)$ are continuous together with the derivatives (according to the assumptions of Theorem 3). Hence, the following limits exist in the region $|q_j^1| \le D^0$

$$\begin{gathered} |a_{ij}(q^1)| \le d, \quad |\partial a_{ij}(q^1)/\partial q_p| \le d, \\ i, j, p = 1, 2, \ldots, N, \quad d = \text{const} \ge 0 \end{gathered} \tag{6.20}$$

We will consider the following function (Fig. 2) as the required function $q^* = q^*(t)$ in inequalities (6.18)

$$q^*(t) = q^1(\gamma t), \quad \gamma = \text{const} > 0, \quad 0 \le t \le \tau, \quad \tau = \tau^1/\gamma \tag{6.21}$$

By construction, the function $q^*(t)$, like $q^1 = q^1(t)$, is changed in the region (6.19) of the form $|q_j^*(t)| \le D^2$. Hence, for $q^* = q^*(t)$ limits of the form (6.20) hold. Then inequalities (6.18) follow from inequalities of the form

$$\sum_{j=1}^{N} d|\ddot{q}_j^*(t)| + \sum_{j,p=1}^{N} \left[d + \frac{1}{2}d \right] |\dot{q}_p^*(t)||\dot{q}_j^*(t)| \le H_i^0, \quad 0 \le t \le \tau \tag{6.22}$$

The following expressions hold for the derivatives $\dot{q}_j^*(t)$.

$$\dot{q}_j^*(t) = \gamma \frac{d}{d\theta}(q_j^1(\theta)), \quad |\dot{q}_j^*(t)| \le \gamma D^1, \quad 0 \le t \le \tau \tag{6.23}$$

(we have taken inequalities (6.19) into account). Similarly

$$|\ddot{q}_j^*(t)| \le \gamma^2 D^2, \quad 0 \le t \le \tau \tag{6.24}$$

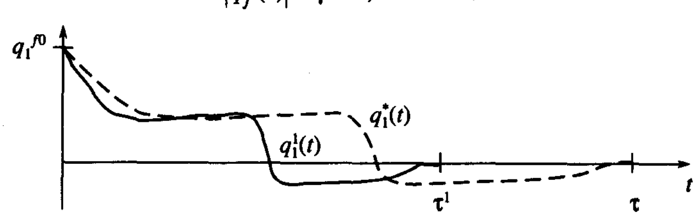

Fig. 2

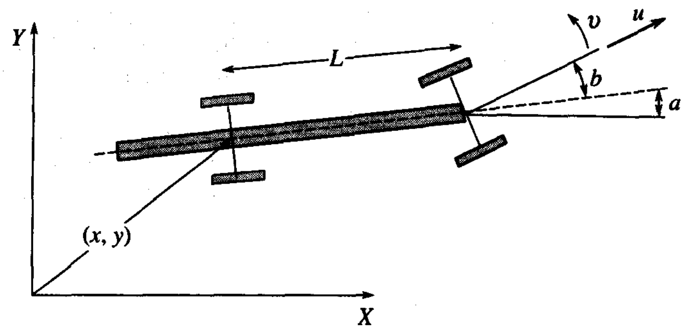

Fig. 3

Taking relations (6.23) and (6.24) into account, the system of inequalities (6.22) follows from the system

$$\sum_{j=1}^{m} d\gamma^2 D^2 + \sum_{j,p=1}^{m} \left[d + \frac{1}{2}d\right](\gamma D^1)^2 \le H_i^0$$

These inequalities hold if the number $\gamma > 0$ is chosen to be sufficiently small. Consequently, the function $q^*(t)$ satisfies the system of inequalities (6.18). In other words, $q^*(t)$ is the motion of system (6.10) (a corresponding control from the class u exists).

Note, finally, that the function $q^*(t)$ in (6.21), as also $q^1(t)$, satisfies the initial conditions (6.17), since

$$q^*(0) = q^1(0), \quad q^*(\tau) = q^1(\gamma\tau) = q^1(\tau^1) = 0$$

Moreover, the function $q^*(t)$ is the solution of the subsystem of constraints (1.2) of system (6.10), i.e.

$$z_s(t) \overset{\text{def}}{=} \sum_{i=1}^{N} f_{si}(q^*(t))\frac{d}{dt}(q_i^*(t)) \equiv 0, \quad 0 \le t \le \tau \tag{6.25}$$

Identities (6.25) are established from the relations

$$z_s\left(\frac{\theta}{\gamma}\right) = \sum_{i=1}^{N} f_{si}(q^1(\theta))\gamma\frac{d}{d\theta}(q_i^1(\theta)), \quad \theta = \gamma t \tag{6.26}$$

That is, from Eqs (6.26) we have

$$z_s(\theta/\gamma) = 0, \quad 0 \le \theta \le \tau^1$$

i.e. Eqs (6.25) follow. Hence we obtain the assertion.

7. AN EXAMPLE OF A CONTROLLABLE NON-HOLONOMIC SYSTEM

To demonstrate the usefulness of the formal assertions presented above from the applied point of view, we will consider an example of a mechanical system (Fig. 3). We will establish that is controllable.

The system shown in Fig. 3 is the simplest model of a wheeled means of motion (like an automobile). It contains a chassis and a rear bridge, and also a controllable front driving bridge. The inertial properties of the system are determined by the mass m and the moment of inertia J of the chassis and the rear bridge, and also by the moment of inertia I of the front bridge. The position of the front bridge is characterized by the angle b and is determined by the torque v. The front wheels are the driving wheels, i.e. a force u is applied to the front bridge. The quantities u and v are the controls.

The equations of motion of this mechanical system can be written in the form of Lagrange equations of the first kind

$$\begin{aligned} m\ddot{x} &= u\cos(a+b) + A\sin a + B\cos a \operatorname{tg} b \\ m\ddot{y} &= u\sin(a+b) - A\cos a + B\sin a \operatorname{tg} b \\ J\ddot{a} &= Lu\sin b - BL, \quad I\ddot{b} = v \end{aligned} \tag{7.1}$$

$$f \overset{\text{def}}{=} \dot{x}\sin a - \dot{y}\cos a = 0, \quad F \overset{\text{def}}{=} \dot{x}\cos a \operatorname{tg} b + \dot{y}\sin a \operatorname{tg} b - L\dot{a} = 0 \tag{7.2}$$

It is assumed that the system contains two constraints (7.2). The first relation of (7.2) reflects the assumption that the rear wheels do not slip in a direction along the axes of wheels, and the second reflects the similar assumption for the front wheels. A and B are Lagrange multipliers. To simplify the analysis we will assume here that the front bridge and the chassis have only a slight effect on one another (the mass and moment of inertia of the front bridge are small). It is also assumed that the centre of mass of the chassis is at the point (x, y), and J is the moment of inertia of the chassis about this point.

An analogue of the above-mentioned property of attainability holds for the subsystem of constraints (7.2) of system (7.1), (7.2).

Theorem A. For arbitrary points (x^f, y^f, a^f) and (x^e, y^e, a^e) of the region $\{x, y, a\}$ of state space $\{x, y, a, b\}$ of the subsystem of constraints (7.2), there is a certain trajectory $\{x^*(t), \dots, b^*(t)\}$, and also a finite instant of time τ such that

$$x^*(0) = x^f, \dots, a^*(0) = a^f, \quad x^*(\tau) = x^e, \dots, a^*(\tau) = a^e \tag{7.3}$$

$$|\dot{x}^*(t)| \le \alpha, \dots, |\dot{b}^*(t)| \le \alpha, \quad |b^*(t)| \le \beta, \quad 0 \le t \le \tau \tag{7.4}$$

where $\alpha > 0$ and $0 < \beta < \pi/2$ are arbitrary numbers specified in advance.

In other words, the analogue of the property of attainability must be justified here, and not simply assumed, as was done above.

The following analogue of Theorem 1 also holds.

Theorem B. Suppose arbitrary constant H_i, which satisfy the condition

$$H_1 > 0, \quad H_2 > 0 \tag{7.5}$$

are specified. Then system (7.1)–(7.2) is controllable in the region $\{x, y, a, \dot{x}, \dot{y}, \dot{a}\}$ of its phase space $\{x, y, a, b, \dot{x}, \dot{y}, \dot{a}, \dot{b}\}$ in the class of permissible controls of the form

$$|u(t)| \le H_1, \quad |v(t)| \le H_2 \tag{7.6}$$

Remark. In Theorems A and B we do not consider the entire phase space of system (7.1), (7.2), but only that part of it which does not include the variable b. This is due to the following factors. In the problem of controlling a wheeled system (Fig. 3), the position b of the steering column essentially plays the role of the control. It is sufficient to change the variable b solely in a limited interval, for example, in the region $|b| < \pi/2$. Hence, from the applied point of view, it makes no sense to investigate the controllability of system (7.1), (7.2) over the whole phase space of the system. Note also that the conditions of Theorem 1 are also not satisfied for system (7.1), (7.2). In particular, the equations of the constraints (7.2) are not continuous, for example, when $b = \pi/2$. This is the formal reason which determines the formulation of Theorems A and B.

Proof of Theorem A. In Theorem A the topic of discussion is the analogue of the property of attainability for the system of constraints (7.2). This property is essentially related to the property of controllability for controllable systems. Thus, we write system (7.2) in the form (6.13)

$$\dot{x} = \dot{\pi}\cos a, \quad \dot{y} = \dot{\pi}\sin a, \quad \dot{a} = \dot{\pi}\operatorname{tg} b/L \tag{7.7}$$

In system (7.7) the quantities $\dot{\pi}$ and b will be formally considered as the control parameters (the controls). If system (7.7) can be transferred from the point $s^f = (x^f, y^f, a^f)$ to the point $s^e = (x^e, y^e, a^e)$ as a result of certain control $\dot{\pi}$ and b, the point s^e of system (7.2) will be attainable from the point s^f.

Lemma A. System (7.7) is controllable in its state space $\{x, y, a\}$ in the class of permissible controls $\dot{\pi}(t)$ and $b(t)$ of the form

$$|\dot{\pi}| \le \gamma_1, \quad |\ddot{\pi}| \le \gamma_2, \quad |b| \le \beta_1, \quad |\dot{b}| \le \beta_2, \quad |\ddot{b}| \le \beta_3 \tag{7.8}$$

where γ_p and β_p are arbitrary specified constants, which satisfy the condition

$$\gamma_p > 0, \quad \beta_p > 0, \quad \beta_1 < \pi/2 \tag{7.9}$$

The proof of Lemma A is given below.

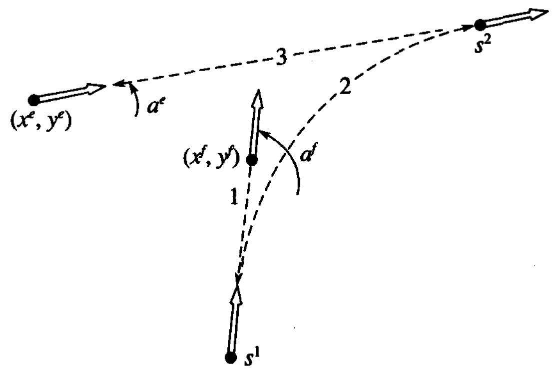

Fig. 4

Obviously, number γ_p and β_p in inequalities (7.9) exist such that inequalities (7.4) hold on the motion of system (7.7). Hence, it follows from Lemma A that the analogue of the property of attainability will hold for system (7.2), which is confirmed in Theorem A.

The introduction of the class of controls (7.8) is due to the use of the property of controllability of system (7.7) to prove Theorem A. Thus, the properties of the controls $\dot{\pi}(t)$ and $b(t)$, obtained in Lemma A, must be taken into account henceforth when analysing the initial system (7.1), (7.2). In particular, the function $\dot{\pi}(t)$ and $b(t)$ must satisfy these equations, and so their derivatives $\ddot{\pi}$ and $\dot{b}$ must be bounded. Here we have also borne in mind constraints (7.6), imposed on the initial controls u and v. Hence, the quantities $\dot{\pi}$ and b in relations (7.8) are assumed to be bounded.

Proof of Lemma A. The motion of system (7.7) from the point s^f to the point s^e consists of three basic steps: (1) the system is displaced from the initial point s^f along the straight line 1 to a certain point s^1, (2) the system essentially turns by a specified angle a^e and is incident on the point s^2, and (3) the system is displaced along straight line 3 to the final points s^e.

An example of the trajectory of motion of the system from the initial point $s^f = (x^f, y^f, a^f)$ to the specified points $s^e = (x^e, y^e, a^e)$ is shown in Fig. 4.

Assertion A_1. No matter how the points s^f and s^1 are arranged on the straight line 1, there are always permissible control $\dot{\pi}_1(t)$, $b_1(t)$ which transfer system (7.7) from s^f to s^1 in a certain finite time t_1.

A similar assertion also holds for the step when the system is displaced from the point s^2 to the point s^e (Fig. 4). Hence, the purpose of the second step in displacing system (7.7) along the curve 2 is essentially the transfer of the system from point s^1 to some point s^2 along the straight line 3.

Assertion A_2. No matter what the straight lines 1 and 3 are, points s^1 and s^2 will always exist on them, and there will also be permissible controls $b_2(t)$ and $\dot{\pi}_2(t)$ which transfer system (7.7) from s^1 to s^2 in a certain finite time t_2.

Hence, for the straight lines 1 and 3 we initially construct the points s^1 and s^2. Only then will system (7.7) transfer from s^f to s^1, then to s^2 and finally to s^e. Consequently, the assertion of Lemma A that it is possible so shift the system from s^f to s^e follows from Assertions A_1 and A_2.

Assertions A_1 and A_2 are proved in the Appendix.

The scheme of the proof of Theorem B. The proof follows the proof of Theorem 3 given above. We have established that system (7.1), (7.2) can be completely slowed down and shifted to the coordinate plane. Moreover, system (7.1), (7.2) can be displaced from one point of the coordinate plane to another (taking into account the assertion of Theorem A).

For stopping, system (7.1), (7.2) is written in the form of a system of Appell equations with quasi-velocities $\dot{\pi} = \dot{x}\cos a + \dot{y}\sin a$ and $\dot{b}$ [5, 6]. This system contains Eq. (7.7), and also the equations

$$\ddot{\pi}\left\{m + \frac{J\,\mathrm{tg}^2 b}{L^2}\right\} = -J\dot{\pi}\dot{b}\frac{\mathrm{tg}\,b}{L^2\cos^2 b} + \frac{u}{\cos b}, \quad I\ddot{b} = v \tag{7.10}$$

By means of a control of the form

$$v = -H_2\,\mathrm{sign}(\dot{b} + e\sqrt{|b|}\,\mathrm{sign}\,b), \quad e = \mathrm{const} > 0$$

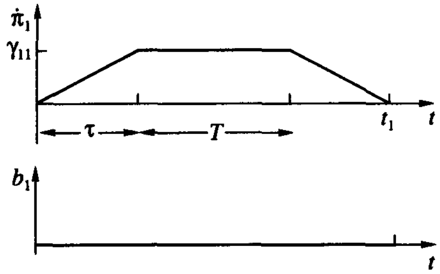

Fig. 5

system (7.7), (7.10) is led into a mode of motion of the form $b(t) = 0$ [10]. In this case the first equation of system (7.10) takes the form $\ddot{\pi}\{m\} = u$. By means of the control $u = -H_1$ sign $(\dot{\pi})$, system (7.7), (7.10) is completely stopped, i.e. the condition $\dot{\pi} = 0$ is guaranteed (also in a finite time).

Note that the stopping of the system described is achieved in a different way compared with the general case (Theorem 3). The point is that Eqs (7.1) and (7.2) are not a special case of systems of the general form (5.1) derived above. Thus, there is a considerable deficit of controls in the system (only one control u occurs in the first three equations of system (7.1), (7.2)). Moreover, the control u occurs in the equations with coefficients which can vanish.

For displacement in the coordinate plane, system (7.1), (7.2) can be written in the form of Maggi's equations

$$m(\ddot{x}\cos a + \ddot{y}\sin a) + J\ddot{a}\,\mathrm{tg}\,b/L = u/\cos b, \quad I\ddot{b} = v, \quad f = 0, \quad F = 0 \tag{7.11}$$

To construct Eqs (7.11) we can take into account the similar transformation in the proof of Lemma 2. In this connection, we note that the equations of the constraints (7.1) can be written in the expanded form (6.13), i.e.

$$f = 0, \quad F = 0, \quad \dot{x}\cos a + \dot{y}\sin a = \dot{\pi} \tag{7.12}$$

Note also that Eqs (7.10) and (7.11), in particular, are constructed on the basis of Eqs (7.12). We note finally that system (7.1), (7.2) transforms into system (7.11) when the factors A and B are eliminated.

Further, for system (7.11) we can establish the existence of the motion $\{x^*(t), y^*(t), a^*(t), b^*(t)\}$ from the point (x^f, y^f, a^f) to the point (x^e, y^e, a^e), which the corresponding controls allow:

$$|u^*(t)| \le H_1, \quad |v^*(t)| \le H_2$$

We have taken into account here the assertion of Theorem A. In particular, the corresponding numbers $\alpha > 0$, $\beta > 0$ in inequalities (7.4) are defined for the specified numbers $H_i > 0$ under conditions (7.5) of Theorem B.

8. APPENDIX

Proof of Assertion A_1. The motion of system (7.7) will occur along a straight line from the specified points s^f to the specified point s^1, if we use controls of the form $\dot{\pi}_1(t)$, $(b)_1(t)$, shown in Fig. 5.

In fact, in this case we have from Eqs (7.7)

$$a(t) = \text{const} \tag{8.1}$$

$$x(t) = x(0) + \cos a\Pi(t), \quad y(t) = y(0) + \sin a\Pi(t), \quad \Pi(t) = \int_0^t \dot{\pi}(y)dy \tag{8.2}$$

For the points s^f and s^1 (Fig. 4) relations (8.2) take the form

$$\begin{gathered} x^f = x^1 + \cos a^f \Pi_1(t_1), \quad y^f = y^1 + \sin a^f \Pi_1(t_1), \\ \Pi_1(t) = \int_0^t \dot{\pi}_1(y)dy; \quad t_1 = T + 2\tau \end{gathered} \tag{8.3}$$

For the quantity $\Pi_1(T + 2\tau)$, which satisfies (8.3), to exist, we will assume that

$$\cos a^f \neq 0, \qquad \sin a^f \neq 0$$

This condition can always be ensured by rotating the system of coordinates.

The following relation holds for the control $\dot{\pi}_1$ in Fig. 5

$$\Pi_1(T+2\tau) = \gamma_{11}(\tau+T), \quad \gamma_{11} = \text{const} \geq 0$$

Hence, relations (8.3) take the form

$$x^f = x^1 + \cos a^f \gamma_{11}(\tau+T), \quad y^f = y^1 + \sin a^f \gamma_{11}(\tau+T) \tag{8.4}$$

We will require that the numbers γ_{11}, τ and T additionally satisfy the conditions

$$|\gamma_{11}| \leq \gamma_1, \quad |\gamma_{11}/\tau| \leq \gamma_2 \tag{8.5}$$

In this case the control $\dot{\pi}_1$ will satisfy the admissibility conditions (7.8) of the controls (the control $b_1(t) = 0$ satisfies these conditions). It is obvious that the positive numbers γ_{11}, τ and T, which satisfy system (8.5), exist. This means that the permissible control $\dot{\pi}_1(t)$ and $b_1(t)$ also exist.

In the above discussions we implied that

$$b_1(0) = 0, \quad \dot{\pi}_1(0) = 0 \tag{8.6}$$

In other words, we assumed that $\dot{\pi}^f = 0$, $b^f = 0$, where $\dot{\pi}^f$, b^f are the initial values of the controls at the initial point $s^f = (x^f, y^f, a^f)$. Note that these conditions can be realized in the initial system (7.1), (7.2). In fact, according to the proof of Theorem B, system (7.1), (7.2) is completely stopped only when the displacement step in the coordinate plane is carried out (Theorem A and Lemma A_1). Hence, the condition $\dot{\pi}^f = 0$ will be satisfied. The condition $b^f = 0$ may also be realized. Thus, the variable b satisfies the fourth equation of system (7.1), (7.2), which is independent of the other variables of the system. Hence, the position $b = 0$ can always be ensured. In this case the state of system (7.1), (7.2) changes, but this is unimportant for the proof of this assertion.

Proof of Assertion A_2. Suppose we are given two non-parallel arbitrary straight lines 1 and 3. If the initial straight lines 1 and 3 are parallel, system (7.7) is first unrolled (the corresponding controls are similar to the controls $b_2(t)$ and $\dot{\pi}_2(t)$, see below). We will show the points s^1 and s^2 exist on the straight lines (Fig. 4), and that permissible controls $b_2(t)$ and $\dot{\pi}_2(t)$ also exist, which transfer system (7.7) from the point s^1 to the point s^2 in a certain finite time t_2.

Consider the rotation of system (7.7) by a specified angle

$$a^f \to a^e \tag{8.7}$$

where a^f is the value of the angle a at the initial point s^1, and a^e is the value of the angle at the final point s^2 (Fig. 4). According to the last equation of system (7.7) its angular position varies as follows:

$$a(t) = a(0) + \frac{1}{L}\int_0^t \dot{\pi}\,\mathrm{tg}\,b\,dy \tag{8.8}$$

For a^f and a^e this relation takes the form

$$a^e = a^f + \frac{1}{L}\int_0^{t_2} \dot{\pi}_2(t)\,\mathrm{tg}(b_2(t))dt \tag{8.9}$$

where t_2 is the time taken to carry out the manoeuvre (8.7).

We will use the controls shown in Fig. 6 as the control $\dot{\pi}_2(t)$ and $b_2(t)$, which realize the objective (8.7). Here the control b_2 only varies when $\dot{\pi}_2 = 0$, while $\dot{\pi}_2$ varies only when $b_2(t) = \text{const}$. This enables us to simplify the expressions derived below.

Taking the above features into account, relation (8.9) can be written in the forms

$$a^e = a^f + \frac{\mathrm{tg}\,\beta_{11}}{L}\int_0^{t_2} \dot{\pi}_2 dt, \quad a^e = a^f + \frac{\mathrm{tg}\,\beta_{11}}{L}(\gamma_{11}(\tau+T)) \tag{8.10}$$

Note that, according to Fig. 6, it is assumed that $\dot{\pi}_2(0) = 0$, $b_2(0) = 0$. In other words, at the point s^1 the controls $\dot{\pi}_1$ and b_1 take zero values. This is allowed since, according to Assertion A_1, the equalities $\dot{\pi}_1(t_1) = 0$, $b_1(t_1) = 0$ hold and similar equalities for the derivatives.

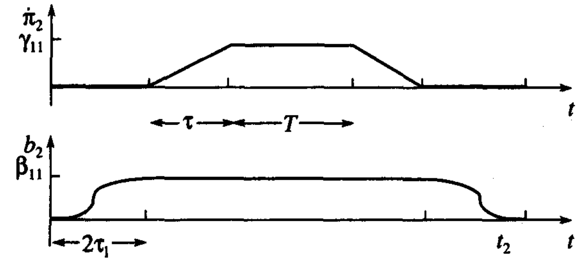

Fig. 6

We will show that the control $\dot{\pi}_2(t)$ and $b_2(t)$ will satisfy the control admissibility conditions (7.8). To do this we seek the control $b_2(t)$ (when $t \in [0, 2\tau_1]$) in the form of a function, the derivative of which has the form

$$\ddot{b}_2(t) = \beta_{31}, \quad t \in [0, \tau_1], \quad \ddot{b}_2(t) = -\beta_{31}, \quad t \in [\tau_1, 2\tau_1], \quad \beta_{31} = \text{const} \geq 0$$

In this case

$$\dot{b}_2(t) = \beta_{31}t, \quad t \in [0, \tau_1], \quad \dot{b}_2(t) = \beta_{31}(2\tau_1 - t), \quad t \in [\tau_1, 2\tau_1]$$

$$b_2(t) = \beta_{31}t^2/2, \quad t \in [0, \tau_1], \quad b_2(t) = \beta_{31}(-\tau_1^2 + 2\tau_1 t - t^2/2), \quad t \in [\tau_1, 2\tau_1]$$

Hence it follows that

$$b_2(2\tau_1) = \beta_{31}\tau_1^2$$

Hence, the last expression in (8.10) takes the final form

$$a^e = a^f + \frac{\operatorname{tg}(\beta_{31}\tau_1^2)}{L}(\gamma_{11}(\tau + T)) \tag{8.11}$$

We will require that the numbers β_{31}, τ_1, γ_{11}, τ and T should additionally satisfy the conditions

$$|\gamma_{11}| = \gamma_1, \quad |\gamma_{11}/\tau| \leq \gamma_2, \quad |\beta_{31}\tau_1^2| \leq \beta_1, \quad |\beta_{31}\tau_1| \leq \beta_2, \quad |\beta_{31}| \leq \beta_3 \tag{8.12}$$

In this case the controls $\dot{\pi}_2(t)$ and $b_2(t)$ will satisfy the control admissibility conditions (7.8). Obviously, additional numbers β_{31}, τ_1, γ_{11}, τ and $T \geq 0$ exist, which satisfy system (8.11), (8.12). For example, the positive numbers β_{31} and τ_1 can only be chosen from conditions (8.12). In this case the positive numbers γ_{11}. τ and $T \geq 0$ ensure that the remaining conditions are satisfied.

This means that permissible control $\dot{\pi}_2(t)$ and $b_2(t)$ exist for which system (7.7) is unrolled by an angle a^e, according to relation (8.7). The control $\dot{\pi}_2(t)$ and $b_2(t)$ considered essentially solve the initial problem of transferring system (7.7) from the point s^1 to the point s^2 in a certain finite time (Fig. 4). In fact, we will denote by

$$\Delta x = x^1 - x^2, \quad \Delta y = y^1 - y^2$$

the change in the coordinates of system (7.7) in the time t_2 of its turning (8.9). The vector

$$e = (\Delta x, \Delta y)$$

then defines the coordinates x^1, y^1 of the required position of the point $s^1 = (x^1, y^1, a^f)$ on straight line 1 (Fig. 4). Assertion A_2 is proved.

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